Thickness Effect on the Force of Slender Delta Wings in Hypersonic Flow

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Newtonian flow theory for a conical flow field has been applied to estimate the thickness effect on the force of slender delta wings at moderate angle of attack. Assuming the shock shape in a cross plane perpendicular to the wing, we have obtained the pressure coefficient for wings with either diamond or lens-type cross section. Comparison with both other theoretical and experimental works has been made. It can be concluded that the thickness of a wing diminishes the normal force for smaller values of the parameter Ω but increases it for larger values, where Ω is a similarity parameter introduced by Messiter.

I. Introduction

THE Newtonian theory for hypersonic flow of an inviscid gas has been developed by the author,[1] Cole and Brainerd,[2] and Messiter,[3] for axisymmetrical, two-dimensional, and conical flows, respectively. The latter two references dealt with the flow past a flat delta wing at a very high and a moderate angle of attack, respectively.

In this paper, the force on a slender delta wing with finite thickness at a moderate angle of attack is discussed within the frame of a conical flow theory. The further assumption that a shock wave is detached from the leading edge of the wing imposes some restriction on the magnitude of aspect ratio. The case with a very high angle of attack will be discussed later in a separate paper.

All the analysis can be made in a cross plane perpendicular to the center line of a wing due to the assumption of a conical flow. The inverse method that the shape of a shock is assumed with a few adjustable parameters has been used exclusively.

Some general results, independent of the individual wing shape, such as the shock stand-off distance and the velocity and the pressure distribution along the axis of symmetry in a cross plane, have been obtained.

The interesting fact that the shape of a shock is not sensitive to the change of the body shape is exemplified by the fact that the first two terms of the expansion of a shock wave are required to satisfy the boundary condition at the body, even if we retain only the lowest-order term.

Two-term expansion applied to the flat delta wing gives a good agreement with the Messiter's results, which were obtained by solving a complicated functional equation with the aid of a digital computer.

Application has been made to the slender delta wings with either the diamond or the parabolic-arc (lens-type) cross section. It has been found that the thickness of a wing diminishes the pressure force for smaller values of Ω but increases for larger values, where Ω is a similarity parameter introduced by Messiter.[4]

Finally, we shall briefly comment on the other works for a conical flow theory. In a recent paper, Kennet[5] attacked the flat wing case by using the Dorodnitsyn-Belotserkovskii method and obtained a result that shows a better agreement with existing experimental data for small Ω. On the other hand, Melnik and Scheuing[6] pointed out in their paper that the conical flow theory suggests the existence of an entropy layer near the surface of a wing, which was completely discarded in the Kennet's paper. Melnik and Scheuing assumed a singular solution for the entropy layer and showed the possibility of matching it with the outer solution. The complete theory with a consideration of the entropy layer is still lacking and is left to future studies.

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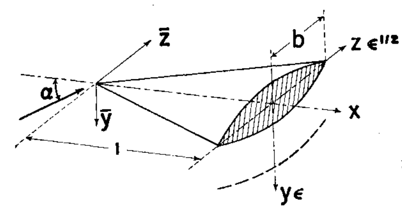

Fig. 1 Coordinate system.

II. Summary of Conical Flow Theory

In this section, we shall summarize the conical flow theory for an inviscid hypersonic flow developed by Messiter. The important parameter for a hypersonic flow is the density ratio across a shock wave and is defined in our case by the relation

$$\epsilon = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1}\frac{1}{M_\infty^2 \sin^2\alpha} \tag{2.1}$$

where γ is the adiabatic index of a gas, M_∞ is the Mach number at infinity upstream, and α is the shock-wave angle. In the Newtonian flow theory, α can be taken also as the angle of attack of a thin wing.

Referring to Fig. 1, we shall introduce the reduced coordinates (x, y^*, z^*) instead of $(\bar{x}, \bar{y}, \bar{z})$ by the relations

$$\left.\begin{aligned} x &= \bar{x} \\ y^* &= \bar{y}/\epsilon \\ z^* &= \bar{z}/\epsilon^{1/2} \end{aligned}\right\} \tag{2.2}$$

Further, we shall assume that the velocity components $(\bar{u}, \bar{v}, \bar{w})$, the pressure P, and the density ρ, can be expanded in the forms

$$\left.\begin{aligned} \bar{u}/\bar{U}_\infty &= \cos\alpha + \epsilon \sin\alpha \cdot u^*(x, y^*, z^*) \\ \bar{v}/\bar{U}_\infty &= \epsilon \sin\alpha \cdot v^*(x, y^*, z^*) \\ \bar{w}/\bar{U}_\infty &= \epsilon^{1/2} \sin\alpha \cdot w^*(x, y^*, z^*) \\ (P - P_\infty)/(\rho_\infty \bar{U}_\infty^2) &= \sin^2\alpha + \epsilon \sin^2\alpha \cdot p^*(x, y^*, z^*) \\ \rho_\infty/\rho &= \epsilon + \epsilon^2 \sigma^*(x, y^*, z^*) \end{aligned}\right\} \tag{2.3}$$

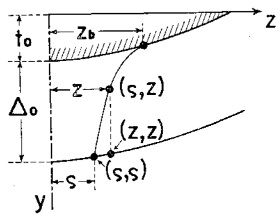

Fig. 2 Configuration in cross plane.

where $\bar{U}_\infty$ is the speed of an oncoming flow, and suffix ∞ refers to the uniform state at infinity.

Justification of this expansion scheme is found in getting a consistent system of equations as well as the boundary conditions on the shock wave, which take the forms in the first approximation

$$\left.\begin{aligned}
&\partial v^*/\partial y^* + \partial w^*/\partial z^* = 0\\
&\cot\alpha\cdot(\partial u^*/\partial x) + v^*(\partial u^*/\partial y^*) + w^*(\partial u^*/\partial z^*) = 0\\
&\cot\alpha\cdot(\partial v^*/\partial x) + v^*(\partial v^*/\partial y^*) + w^*(\partial v^*/\partial z^*) = \\
&\qquad -\partial p^*/\partial y^*\\
&\cot\alpha\cdot(\partial w^*/\partial x) + v^*(\partial w^*/\partial y^*) + w^*(\partial w^*/\partial z^*) = 0\\
&\{\cot\alpha\cdot(\partial/\partial x) + v^*(\partial/\partial y^*) + w^*(\partial/\partial z^*)\}\cdot\\
&\qquad (p^* + \gamma\sigma^*) = 0
\end{aligned}\right\} \quad (2.4)$$

with

$$\left.\begin{aligned}
u_s^* &= -\partial y_s^*/\partial x\\
v_s^* &= (\partial y_s^*/\partial x)\cot\alpha - 1 - (\partial y_s^*/\partial z^*)^2\\
w_s^* &= -\partial y_s^*/\partial z^*\\
p_s^* &= -1 - (\partial y_s^*/\partial z^*)^2 + 2(\partial y_s^*/\partial x)\cot\alpha\\
\sigma_s^* &= [(\gamma+1)/(2+N)]\{(\partial y_s^*/\partial z^*)^2 - \\
&\qquad 2(\partial y_s^*/\partial x)\cot\alpha\}\\
&\text{where}\\
N &= (\gamma - 1)/[1/(M_\infty^2\sin^2\alpha)]
\end{aligned}\right\} \quad (2.5)$$

Equation (2.4) shows that v^*, w^*, and p^* can be solved independently of u^* and σ^*. Hereafter in this paper we shall consider only these three variables.

Consistent with the assumption of a conical flow, we shall introduce the new variables defined by

$$y = y^*/x \qquad z = z^*/x \quad (2.6a)$$

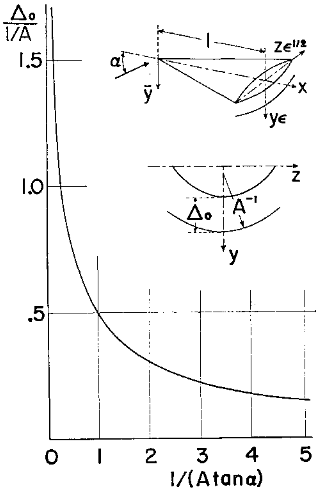

Fig. 3 Standoff distance of shock wave.

$$\left.\begin{aligned}
v^* &= v(y, z;\ \cot\alpha)\\
w^* &= w(y, z;\ \cot\alpha)\\
p^* &= p(y, z;\ \cot\alpha)\\
y_s^* &= xy_s(z)
\end{aligned}\right\} \quad (2.6b)$$

Equation (2.6a) is equivalent to taking the axes in a cross plane perpendicular to the x axis at $x = 1$.

Now, the fundamental equations and the shock boundary conditions are recast in the forms

$$\left.\begin{aligned}
&(\partial v/\partial y) + (\partial w/\partial z) = 0\\
&(v - y\cot\alpha)(\partial v/\partial y) + (w - z\cot\alpha)(\partial v/\partial z) = \\
&\qquad -(\partial p/\partial y)\\
&(v - y\cot\alpha)(\partial w/\partial y) + (w - z\cot\alpha)(\partial w/\partial z) = 0
\end{aligned}\right\} \quad (2.7)$$

and

$$\left.\begin{aligned}
v_s &= (y_s - xy_s')\cot\alpha - 1 - y_s'^2\\
w_s &= -y_s'\\
p_s &= -1 - y_s'^2 + 2(y_s - zy_s')\cot\alpha
\end{aligned}\right\} \quad (2.8)$$

respectively.

Equation (2.7) has two real characteristics such as $z =$ const and $\zeta =$ const, where ζ is defined by

$$(v - y\cot\alpha)(\partial\zeta/\partial y) + (w - z\cot\alpha)(\partial\zeta/\partial z) = 0 \quad (2.9)$$

and is constant on a streamline in a cross plane.

In order to fix the ζ value, we shall supplement the condition (see Fig. 2)

$$\zeta = z \qquad \text{on shock} \quad (2.10)$$

Transforming the independent variables from (y, z) to (ζ, z), we can get the equations

$$(\partial v/\partial\zeta) + (\partial w/\partial z)(\partial y/\partial\zeta) - (\partial w/\partial\zeta)(\partial y/\partial z) = 0 \quad (2.11a)$$

$$(\partial v/\partial z)(\partial y/\partial\zeta)(w - z\cot\alpha) = -(\partial p/\partial\zeta) \quad (2.11b)$$

$$(\partial w/\partial z) = 0 \quad (2.11c)$$

$$(\partial y/\partial z)(w - z\cot\alpha) = v - y\cot\alpha \quad (2.11d)$$

where y should be considered as a dependent variable.

The general solutions of Eq. (2.11) subject to the shock boundary condition (2.8) are easily obtained in the forms

$$y(\zeta, z) = -\int_0^z w(z_1)dz_1 + (\Delta_0 + t_0) - \int_\zeta^z \frac{w(\zeta_1) - z\cot\alpha}{\{w(\zeta_1) - \zeta_1\cot\alpha\}^2}d\zeta_1 \quad (2.12a)$$

$$w(\zeta, z) = w(\zeta) = -y_s'(\zeta) \quad (2.12b)$$

$$\frac{v(\zeta, z) - y(\zeta, z)\cot\alpha}{w(\zeta) - z\cot\alpha} = -w(z) - \frac{1}{w(z) - z\cot\alpha} + \cot\alpha\int_\zeta^z \frac{1}{\{w(\zeta_1) - \zeta_1\cot\alpha\}^2}d\zeta_1 \quad (2.12c)$$

$$p(\zeta, z) = -1 - w^2(z) + 2\left\{\Delta_0 + t_0 - \int_0^z w(z_1)dz_1\right\}\cot\alpha + 2zw(z)\cot\alpha + \left\{-1 + \frac{1}{[w(z) - z\cot\alpha]^2}\right\}w'(z)\int_\zeta^z \frac{[w(\zeta_1) - z\cot\alpha]^3}{[w(\zeta_1) - \zeta_1\cot\alpha]^2}d\zeta_1 \quad (2.12d)$$

where Δ_0 and t_0 denote the shock stand-off distance and the body thickness on the symmetrical axis in a reduced cross plane, respectively.

Equation (2.12) includes an unknown function $w(\zeta)$, which should be determined by the boundary condition on the body surface.

III. Boundary Condition on the Body Surface

In general, the boundary condition on the body can be expressed in a conical flow theory in the form

$$(v - y \cot\alpha) = (w - z \cot\alpha)(dy/dz) \tag{3.1}$$

which expresses no mass flow through a surface.

At first it should be noticed that the body surface cannot be coincident with any particular streamline (ζ = const) in a cross plane, because we can show that the shock stand-off distance would be negative by such assumption. However, within the frame of a conical theory, all of the streamlines in a cross field may terminate and stop on the body, and we shall assume the relations

$$v_b = y_b \cot\alpha \qquad w_b = z_b \cot\alpha \tag{3.2}$$

where the subscript b refers to the value on the body.

Since w is the function of ζ only from (2.12b), the second relation of Eq. (3.2) shows the functional relation between ζ and z_b.

Considering z_b as a function of ζ, we have from Eq. (2.12a)

$$\begin{aligned} y_b(\zeta) &= y[\zeta, z_b(\zeta)] \\ &= -\int_0^{z_b(\zeta)} w(z_1)dz_1 + \Delta_0 + t_0 - \\ &\qquad \int_\zeta^{z_b(\zeta)} \frac{w(\zeta_1) - z_b(\zeta)\cot\alpha}{[w(\zeta_1) - \zeta_1 \cot\alpha]^2}\, d\zeta_1 \end{aligned} \tag{3.3}$$

which holds on the surface of a body.

It follows from Eq. (3.3) that the inclination of a body is given by

$$\begin{aligned} (dy/dz)_{\text{body}} &= (dy_b/d\zeta)/(dz_b/d\zeta) \\ &= -w(z_b) - \frac{1}{w(z_b) - z_b \cot\alpha} + \\ &\qquad \cot\alpha \int_\zeta^{z_b} \frac{d\zeta_1}{[w(\zeta_1) - \zeta_1 \cot\alpha]^2} \end{aligned} \tag{3.4}$$

which agrees with the condition formally obtained from (3.1) and (2.12c).

In this paper, Eq. (3.4) together with the relation

$$w(\zeta) = z_b \cot\alpha \tag{3.5}$$

is used as the fundamental equation to determine w.

The boundary conditions for w are given in the form

$$\left.\begin{aligned} w(0) &= 0 \\ w(z_b) &= z_b \cot\alpha + 1 \end{aligned}\right\} \tag{3.6}$$

Although the former comes from the symmetry of a wing profile in a cross plane, the latter, as was shown by Messiter, comes from the condition that a singularity of the shock curvature occurs at the point corresponding to a wing tip. The second relation can be also interpreted as the physical condition that the circumferential component of the velocity is equal to the local sonic velocity at that point.

Comparing Eq. (3.4) with Eqs. (2.12c) and (2.11d), we can get the relation

$$(dy/dz)_{\text{body}} = (\partial y/\partial z)_{\zeta-\text{const}} \tag{3.7}$$

Equation (3.7) shows that the streamline in a cross plane approaches the body surface in its tangential direction, even if it terminates there.

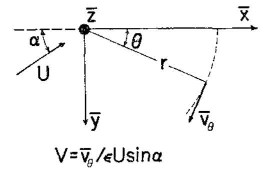

Fig. 4 Physical meaning of V.

IV. Some General Results†

Without relating to any particular body shape, we can get some general results such as the shock stand-off distance and the pressure and velocity distribution on the symmetrical axis in a cross plane.

In the first place, we shall assume that

$$w(\zeta) = A\zeta + O(\zeta^2) \qquad (A > 0) \tag{4.1}$$

where A is the curvature of the shock wave at the nose. Then, by Eq. (3.5) the body can be expressed in the form

$$z_b = A\zeta \tan\alpha + O(\zeta^2) \tag{4.2}$$

Substituting Eq. (4.2) into Eq. (2.12a), we have, after a few manipulations,

$$y(\zeta, z_b) = \Delta_0 + t_0 - \frac{A \log(A \tan\alpha) + \cot\alpha - A}{(A - \cot\alpha)^2} + O(\zeta^2).$$

Taking the limit as ζ tends to zero, we get the stand-off distance referred to the radius of the curvature of the shock in the form

$$\frac{\Delta_0}{(1/A)} = \frac{\log(A \tan\alpha) + 1/(A \tan\alpha) - 1}{\{1 - (1/A \tan\alpha)\}^2} \tag{4.3}$$

In Fig. 3, $\Delta_0/(1/A)$ is plotted as the function of $1/(A \tan\alpha)$.

By the definition of ζ, the value of (z/ζ) becomes unity on the shock wave, whereas from Eq. (4.2) it turns out to be $A \tan\alpha$ at the body on a symmetrical line.

Substituting the relation

$$z = k\zeta$$

with constant k into Eq. (2.12a) and taking the limit as $\zeta \rightarrow 0$, we can get the parametric representation for the pressure and the velocity distribution on the symmetrical axis in the forms

$$\frac{\Delta(k)}{\Delta_0} = \frac{\log(A \tan\alpha/k) + (k/A \tan\alpha) - 1}{\log(A \tan\alpha) + (1/A \tan\alpha) - 1} \tag{4.4a}$$

$$\bar{V}(k) \equiv v(k) - y(k)\cot\alpha = -\frac{1}{k}\frac{(A \tan\alpha - k)^2}{(A \tan\alpha - 1)^2} \tag{4.4b}$$

$$\begin{aligned} \bar{p}(k) &\equiv p(k) + 1 - 2(\Delta_0 + t_0)\cot\alpha = \\ &\frac{A \tan\alpha}{(A \tan\alpha - 1)^4 k^2}\left[(k - 1)\left\{\frac{k+1}{2}(A \tan\alpha)^3 - \right.\right. \\ &\qquad \left.\left. 3k(A \tan\alpha)^2 - k^2\right\} + 3(A \tan\alpha)k^2 \log k\right] \end{aligned} \tag{4.4c}$$

where $\Delta(k)$ is the distance measured along the symmetrical line from the body, and $\bar{V}(k)$ has a simple physical meaning as the circumferential component of the velocity in spherical polar coordinates (see Fig. 4).

In Figs. 5 and 6, $\bar{V}$ and $\bar{p}$ vs (Δ/Δ_0) are shown for various values of $A \tan\alpha$, respectively.

V. Singular Solution

All of the formulas given in Eqs. (4.3) and (4.4) diverge at $A \tan\alpha = 1$. However, it is easily seen that the last equation

† Numerical calculations in this section and in Sec. VII have been done by using the computer at Wright-Patterson Air Force Base with the help of W. Hoelmer, to whom the author's thanks are due.

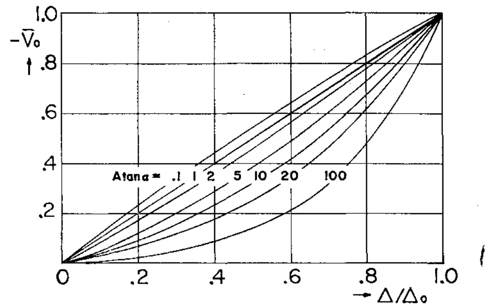

Fig. 5 Velocity distribution along symmetrical axis in cross plane.

of (2.7) is satisfied by

$$w(y, z) = z \cot\alpha \tag{5.1}$$

Substituting this relation into Eq. (2.9), we have ζ = function of z only, which shows that all the streamlines are parallel to y axis. The supplementary condition given in Eq. (2.10) leads us to

$$\zeta = z \tag{5.2}$$

everywhere, and it follows from Eq. (4.2) that this flow pattern corresponds to the exceptional case where

$$A \tan\alpha = 1 \tag{5.3}$$

Taking the limit of Eq. (4.3) as $A \tan\alpha \to 1$, we have

$$\Delta_0 = (1/2A) = \tfrac{1}{2} \tan\alpha \tag{5.4}$$

Equation (2.8) together with Eqs. (5.1) and (5.4) gives

$$y_s - t_0 = \tfrac{1}{2}(\tan\alpha - z^2 \cot\alpha) \tag{5.5}$$

where t_0 is the (reduced) thickness of the cross section of a wing.

From the first equation of (2.7) with the shock condition (2.8), we can get

$$v = -y \cot\alpha - z^2 \cot^2\alpha + 2t_0 \cot\alpha \tag{5.6}$$

Generally, Eq. (5.6) does not satisfy the boundary condition on the body, which is expressed from Eqs. (3.1) and (5.1) in the form $v = y \cot\alpha$.

When and only when the body is expressed in the form

$$y = -(z^2/2) \cot\alpha + t_0 \tag{5.7}$$

the preceding condition is satisfied, and Eqs. (5.1) and (5.6) can be considered as the solution for such special body.

If the aspect ratio of a delta wing is denoted by $2b$, we have from Eq. (5.7)

$$t_0 = (b^2/2\epsilon) \cot\alpha$$

and the thickness ratio of a cross profile is given by $(b/2) \cot\alpha$ (see Fig. 7).

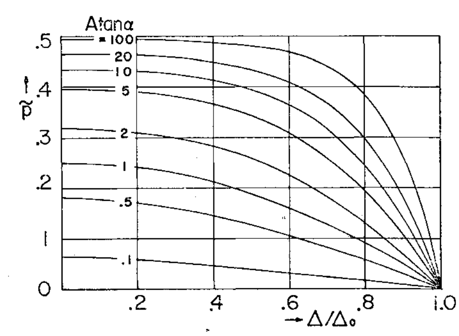

Fig. 6 Reduced pressure distribution along symmetrical axis in cross plane.

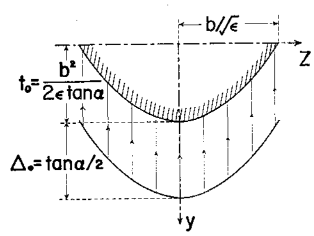

Fig. 7 Singular flow pattern.

The pressure is obtained by the simple quadrature of the second equation of (2.7) with the shock condition (2.8) in the form

$$\tilde{p} = \tfrac{1}{4} - (y - t_0 + \tfrac{1}{2}z^2 \cot\alpha)^2 \cot^2\alpha \tag{5.8}$$

It is clearly seen from Eq. (5.8) that the pressure is constant on the body. In Figs. 5 and 6, the corresponding values to the particular solution are also shown.

VI. Inverse Method

Since the fundamental equation to determine w, Eq. (3.4) with the condition (3.5), is of a very complicated form, we shall have recourse to the inverse method, in which the shock shape is assumed with a few unknown constants to be determined later.

Messiter obtained a functional equation by differentiating Eq. (3.4) with respect to ζ, but this method would miss some of the flow such as that past a wing with a diamond cross section (constant inclination) unless special attention is paid.

In this paper we shall consider exclusively the symmetrical flow and assume

$$w(\zeta) = A_1\zeta \pm A_2\zeta^2 + A_3\zeta^3 + O(\zeta^4) \tag{6.1}$$

where the double sign should be taken plus for $\zeta > 0$ and minus for $\zeta < 0$. Then, the shape of a shock wave is given by

$$y_s = \Delta_0 + t_0 - \tfrac{1}{2}A_1\zeta^2 \mp \tfrac{1}{3}A_2\zeta^3 - \tfrac{1}{4}A_3\zeta^4 + O(\zeta^5) \tag{6.2}$$

By Eq. (3.5), we have

$$z_b = \{A_1\zeta \pm A_2\zeta^2 + A_3\zeta^3 + O(\zeta^4)\} \tan\alpha \tag{6.3}$$

Substituting Eqs. (6.1) and (6.3) into Eq. (3.4), we have

$$\left(\frac{dy}{dz}\right)_{\text{body}} = \pm \left\{\frac{a_1 + 1}{a_1(a_1 - 1)^2} - 2\frac{\log a_1}{(a_1 - 1)^3}\right\} a_2 - \left\{\frac{1}{a_1^3}(a_2^2 - a_1a_3 + a_1^4)\right\} \frac{z}{\tan\alpha} + O(z^2) \tag{6.4}$$

with

$$a_i = A_i(\tan\alpha)^i \qquad (i = 1, 2, 3) \tag{6.5}$$

It is interesting to note that a_1 is accompanied by a_2 or a_3 in the lowest-order term, and we must retain at least two terms for the shock expansion even if we content ourselves with the first approximation.

This shows that the body shape is very sensitive to the change of the shock shape, whereas the shock is not so sensitive to the change of body.

The pressure distribution on the body is calculated from Eq. (2.12d) with Eq. (6.3) in the form

$$p_{\text{body}} = \alpha_1 + 2\left(\frac{a_2}{a_1^2}\right) \sigma \left(\frac{z}{1/A_1}\right) + \left\{\beta + \beta_1\left(\frac{a_3}{a_1^3}\right) + \beta_2\left(\frac{a_2}{a_1^2}\right)^2\right\}\left(\frac{z}{1/A_1}\right)^2 + O(z^3) \tag{6.6}$$

where

$$
\left.\begin{aligned}
\alpha_1 &= 2t_0 \cot\alpha + \left\{-\frac{3}{(a_1-1)^3} - \frac{9}{2}\frac{1}{(a_1-1)^2} - \frac{3}{(a_1-1)} - \frac{1}{2}\right\} + \left\{\frac{3}{(a_1-1)^4} + \frac{6}{(a_1-1)^3} + \frac{5}{(a_1-1)^2} + \frac{2}{(a_1-1)}\right\}\log a_1 \\
\sigma &= \left\{\frac{2}{(a_1-1)^4} + \frac{3}{(a_1-1)^3} + \frac{2}{3}\frac{1}{(a_1-1)^2} - \frac{1}{6}\frac{1}{(a_1-1)} + \frac{1}{6}\right\} - \left\{\frac{2}{(a_1-1)^5} + \frac{4}{(a_1-1)^4} + \frac{2}{(a_1-1)^3}\right\}\log a_1 \\
\beta &= \left\{\frac{3}{(a_1-1)} - \frac{3}{2}\right\} - \frac{3}{(a_1-1)^2}\log a_1 \\
\beta_1 &= \left\{\frac{6}{(a_1-1)^4} + \frac{6}{(a_1-1)^3} - \frac{5}{2}\frac{1}{(a_1-1)^2} - \frac{3}{2}\frac{1}{(a_1-1)} + \frac{3}{4}\right\} - 3\left\{\frac{2}{(a_1-1)^5} + \frac{3}{(a_1-1)^4} + 0 - \frac{1}{(a_1-1)^2}\right\}\log a_1 \\
\beta_2 &= \left\{-\frac{5}{(a_1-1)^5} - \frac{27}{2}\frac{1}{(a_1-1)^4} - \frac{23}{3}\frac{1}{(a_1-1)^3} + \frac{35}{12}\frac{1}{(a_1-1)^2} + \frac{4}{3}\frac{1}{(a_1-1)} - \frac{1}{2}\right\} + \left\{\frac{5}{(a_1-1)^6} + \frac{16}{(a_1-1)^5} + \frac{14}{(a_1-1)^4} + 0 - \frac{3}{(a_1-1)^2}\right\}\log a_1
\end{aligned}\right\} \quad (6.7)
$$

The singularity at $a_1 = 1$ is apparent, and we can get the limiting value of all these coefficients as follows:

$$
\left.\begin{aligned}
&\alpha_1 \to 2t_0 \cot\alpha + \tfrac{1}{4} \\
&\sigma \to \tfrac{1}{10} \\
&\beta \to 0 \qquad \beta_1 \to \tfrac{3}{10} \qquad \beta_2 \to -\tfrac{2}{15}
\end{aligned}\right\} \quad (6.8)
$$

Confining ourselves to a delta wing with aspect ratio $2b$, we shall define the normal force coefficient by

$$
C_N = \frac{\int_0^1 d\bar{x} \int_{-b\bar{x}}^{b\bar{x}} P_{\text{body}} d\bar{z}}{\frac{1}{2}\rho_\infty \bar{U}_\infty^2 \int_0^1 d\bar{x} \int_{-b\bar{x}}^{b\bar{x}} d\bar{z}}
$$

In our notations, this can be transformed into the form

$$
\tilde{C}_N \equiv \frac{C_N - 2\sin^2\alpha - (2/\gamma M_\infty^2)}{\epsilon \sin^2\alpha} = \frac{2}{\Omega}\int_0^\Omega p_{\text{body}}(a_1 t) dt \quad (6.9)
$$

where Ω is the similarity parameter introduced by Messiter[3, 4] and is defined by

$$
\Omega = b/(\epsilon^{1/2} \tan\alpha) \quad (6.10)
$$

Substitution of Eq. (6.6) into Eq. (6.9) gives us

$$
\tilde{C}_N = 2\alpha_1 + 2\left(\frac{a_2}{a_1^2}\right)\sigma(a_1\Omega) + \frac{2}{3}\left\{\beta + \beta_1\left(\frac{a_3}{a_1^3}\right) + \beta_2\left(\frac{a_2}{a_1^2}\right)^2\right\}(a_1\Omega)^2 + \ldots \quad (6.11)
$$

The unknown constants $a_1, a_2, a_3, \ldots$ are determined by specifying the cross-sectional form of a wing, and we shall discuss the thickness effect in the next section.

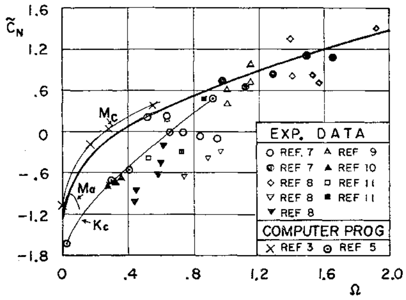

Fig. 8 Comparison with other theoretical and experimental data for flat delta wing; M_a = analytical results by Messiter,[3] M_c = computer program by Messiter,[3] K_c = computer program by Kennet.[5]

VII. Examples

In this section, we shall discuss a few flow examples past a delta wing with thickness by making use of the three-term expansion for the shock shape. In order to check the accuracy of this method, we shall start from the case of a flat wing.

Flat Wing

A flat wing is characterized by

$$
(dy/dz)_{\text{body}} = 0 \qquad h = 0
$$

and we can get from Eq. (6.4)

$$
a_2 = 0 \qquad a_3 = a_1^3 \quad (7.1)
$$

Substituting these relations into Eqs. (6.6) and (6.11), we have

$$
\left.\begin{aligned}
p_{\text{body}} &= \alpha_1 + (\beta + \beta_1)[z/(1/A_1)]^2 \\
\tilde{C}_N &= 2\alpha_1 + \tfrac{2}{3}(\beta + \beta_1)(a_1\Omega)^2
\end{aligned}\right\} \quad (7.2)
$$

The unknown constant a_1 can be determined from Eq. (3.6), which is reduced to the cubic equation

$$
(a_1\Omega)^3 + (a_1\Omega) = \Omega + 1 \quad (7.3)
$$

In Fig. 8, the reduced pressure coefficient $\tilde{C}_N$ is shown by the heavy line curve for the range (0, 2.0) of Ω. In this figure are also drawn the Messiter's analytical results[3] by either digital computer program or analytical formula with a few existing experimental data[7-11] reproduced from his paper. Kennet's[5] result, commented on at the end of the Introduction, is drawn by a thin line, too. It can be seen that the present simple method is in a good agreement with the Messiter's result by a computer program and with the experimental data for the range $0.8 < \Omega < 2.0$.

It may be interesting to note that the streamlines near the axis of symmetry in a cross field come closer on the body for $a_1 < 1$ and vice versa, which is seen from Eq. (4.2). It is seen from Eq. (7.3) that a_1 decreases monotonously with increasing Ω and becomes unity at $\Omega = 1$ (see Fig. 9).

For a larger value of Ω, the shock wave is expected to attach to the leading edge of wing. Babaev studied such a flow in his recent paper[12] and showed the existence of a central region where the streamlines converge. Thus, we can conclude that the flow patterns in Fig. 9 correspond, at least qualitatively, to the transition from the flow with a detached shock wave to that with an attached shock wave.

Lens-Type Cross Section

Let the cross section be expressed in the form

$$
\bar{y} = -(h/b^2)\bar{z}^2 + h \quad (7.4)
$$

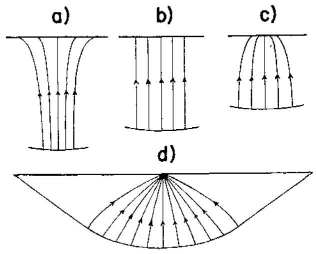

Fig. 9 Streamlines near symmetrical axis: a) $r < 1$, b) $r = 1$, c) $r < 1$, and d) flow with attached shock.[12]

where h and b refer to the thickness and the breadth of a cross section at $\bar{x} = 1$, respectively.

Differentiating Eq. (7.4) with respect to $\bar{z}$, we have, in the reduced notations,

$$(dy/dz)_{\text{body}} = -(2C/\Omega)(z/\tan\alpha) \tag{7.5}$$

where C is the parameter relating to the thickness ratio and is defined by

$$C = h/(b\epsilon^{1/2}) \tag{7.6}$$

Comparing Eq. (7.5) with Eq. (6.4), we have

$$\left.\begin{aligned} a_2 &= 0 \\ a_3/a_1^3 &= 1 - 2C/(a_1\Omega) \end{aligned}\right\} \tag{7.7}$$

Substituting these relations into Eq. (6.11), we have

$$\tilde{C}_N = 2\alpha_1 - \tfrac{4}{3}C\beta_1(a_1\Omega) + \tfrac{2}{3}(\beta + \beta_1)(a_1\Omega)^2 \tag{7.8}$$

The condition to determine a_1, is obtained from Eq. (3.6) together with the relation (7.7) in the form

$$(a_1\Omega)^3 - 2C(a_1\Omega)^2 + (a_1\Omega) = \Omega + 1 \tag{7.9}$$

Figure 10 shows the typical curves of $\tilde{C}_N$ against Ω for $C = 0, 0.5, 1.0, 1.56$. The curve labeled $C = 0$ corresponds to a flat wing, and it can be seen clearly that the thickness effect decreases the force on a wing for small Ω but increases for large Ω.

Diamond Cross Section

Assuming that the cross section is given by

$$(\bar{y}/h) \pm (\bar{z}/b) = 1 \tag{7.10}$$

we have by simple differentiation

$$(dy/dz)_{\text{body}} = \mp C \tag{7.11}$$

where C is the shape parameter defined by Eq. (7.6), and the upper and lower sign should be taken for $\bar{z} > 0$ and $\bar{z} < 0$, respectively.

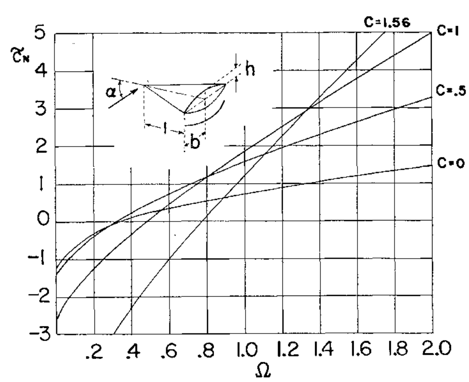

Fig. 10 Reduced pressure coefficient for wing with lens-typed cross section.

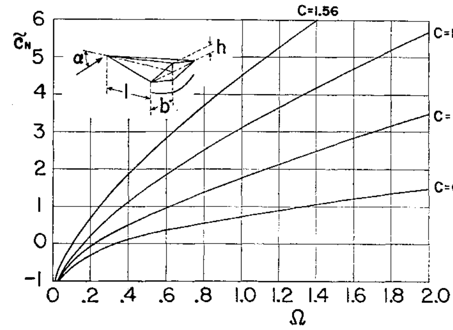

Fig. 11 Reduced pressure coefficient for wing with diamond cross section.

Comparison of Eq. (7.11) with Eq. (6.4) leads us to the relations

$$\frac{a_2}{a_1^2} = \pm C \frac{(a_1 - 1)^3}{a_1\{2a_1 \log a_1 - (a_1^2 - 1)\}}$$

$$\frac{a_3}{a_1^3} = 1 + \left(\frac{a_2}{a_1^2}\right)^2$$

The reduced pressure coefficient is given from Eq. (6.11) in the form

$$\tilde{C}_N = 2\alpha_1 + 2\left(\frac{a_2}{a_1^2}\right)\sigma(a_1\Omega) + \frac{2}{3}\left\{(\beta + \beta_1) + (\beta_1 + \beta_2)\left(\frac{a_2}{a_1^2}\right)^2\right\}(a_1\Omega)^2 \tag{7.12}$$

where a_1 is the positive root of the cubic equation obtained from Eq. (3.6):

$$a_1\Omega \pm \frac{a_2}{a_1^2}(a_1\Omega)^2 + \frac{a_3}{a_1^3}(a_1\Omega)^3 = \Omega + 1 \tag{7.13}$$

In Fig. 11 are shown the typical curves of $\tilde{C}_N$ vs Ω for four values of C: 0, 0.5, 1.0, 1.56. It is clearly seen that the thickness of a wing increases the pressure on a wing for most values of Ω, although it still decreases for very small Ω.

VIII. Conclusions

1) The inverse method, in which the shape of a shock wave is expressed in a three-term expansion, has been applied to the first order theory of the Newtonian flow.

2) Justification of this method is obtained by comparing the result of a flat wing with the Messiter's result by a computer program.

3) Sensitivity of a body shape to the change of a shock shape is exemplified by showing that the lowest-order term of the body is influenced by the first two terms of the shock.

4) The pressure coefficient of a slender delta wing with either a lens-type or a diamond cross section of various thickness has been given in Figs. 10 and 11.

5) For small values of the similarity parameter Ω, the normal force decreases with increasing thickness, whereas for large values of Ω, the former increases with the latter. This effect of the thickness is more eminent for the diamond-type cross section than for the lens-type one.

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Hypersonic Flow over an Oscillating Wedge

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The equations of hypersonic small-disturbance theory are perturbed for small oscillations of a thin, semi-infinite, flexible wedge. A solution for arbitrary smooth mode shapes is found in terms of infinite series. These series express the effects of acoustic waves, generated by the unsteady motion, that propagate in the flow field between the bow shock wave and the wedge surface. They reflect from the wedge surface and the shock wave, but they are attenuated only at the shock wave. Thus the degree of attenuation determines the influence of the waves. This attenuation decreases as the shock strength increases. It is then shown that the results from more approximate methods, such as piston theory and others that attempt to account for a strong bow shock, can be reproduced by successively neglecting terms in the "complete" solution. It appears that the most significant contribution of the complete solution is to change the phase shift of the unsteady pressure. Finally, it is shown that the complete solution is the only one that will produce an adequate representation in the double limit of very large Mach number and adiabatic exponent near unity.

Introduction

Hypersonic flow is distinguished by the fact that the governing equations are nonlinear even for the flow over slender bodies. From the viewpoint of unsteady aerodynamics, this means that the body cannot be idealized to a surface of zero thickness; it will create a shock wave whose effects on the ensuing flow must be considered.

Piston theory has been widely used as a first step in approximating such effects. The theoretical basis for piston theory has been known for many years, but it is generally accepted that Lighthill[1] was the first to recognize its utility. The theoretical and practical limitations on its applicability to aeroelastic problems have been studied by Ashley and Zartarian,[2] among others. The basis for using a piston formula in hypersonic flow is provided by the unsteady analogy of hypersonic small-disturbance theory.[3, 4] It is found that the reduced equations for this theory will transform directly to those for unsteady flow in the transverse plane. For example, we may replace a two-dimensional steady or unsteady problem by that of a one-dimensional piston moving in a tube filled with a gas. The velocity of the piston is identified with the velocity imparted to the fluid by the body. If the original problem is unsteady, there is an additional term due to the motion of the body. However, the character of the piston problem is unchanged, and so in principle there is no difference between steady and unsteady flow in hypersonic small-disturbance theory. In general, the piston will generate at least one shock wave, which corresponds to the bow shock wave.

The problem is greatly simplified by assuming that the piston generates only acoustic waves, for then there exists a very simple relation between the velocity of the piston and the pressure on the face of the piston which corresponds to the surface pressure. The pressure thus calculated gives excellent results provided a number of conditions are met; the details are in Refs. 1 and 2. Physically, this approximation does not take into account three phenomena: the effect of the bow shock wave on the flow, the effect of any interaction between subsequent waves and the bow shock wave, and the effect of waves stronger than acoustic generated by the unsteady motion of the body. For most purposes, the third restriction is not serious, since it is equivalent to linearization in the amplitude of the unsteady motion; indeed, this is all

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